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On the ends of pairs of groups

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Abstract

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We develop a technique for calculating the ends of a pair of groups (G, H) for special types of (G, H).

The concepts of ends of spaces and of groups are described in [2], [4], [5], and [6]. We recall the definitions. Let G be a finitely generated infinite group and let H be a subgroup of G. Let $\mathbb{Z}[G/H]$ (resp. $\overline{\mathbb{Z}[G/H]}$) denote the left G-module of finite (resp. unrestricted) maps of left cosets of H in G into $\mathbb{Z}$ and let $E[G/H]$ be the quotient of $\overline{\mathbb{Z}[G/H]}$ by $\mathbb{Z}[G/H]$. If the index of H in G is infinite, we have an exact sequence

$$0 \to \mathbb{Z} \to H^0(G; E[GH]) \to H^1(G; \mathbb{Z}[G/H]) \to H^1(G; \overline{\mathbb{Z}[G/H]}) \to \cdots .$$

By Shapiro's Lemma, the last term in the above sequence is isomorphic to $H^1(H; \mathbb{Z})$ and thus we obtain the exact sequence:

$$(*) \qquad 0 \to \mathbb{Z} \to H^1(G; E[G/H]) \to H^1(G; \mathbb{Z}[G/H]) \to H^1(H; \mathbb{Z}) \to \cdots .$$

If $p : \tilde{K} \to K$ is a regular cover with K finite and G the group of covering translations of p, then the number of ends of $\tilde{K}/H$ is equal to the rank of $H^0(G; E[G/H])$, which is known to be free abelian. This rank is by definition the number of ends of the pair (G, H) and is denoted $e(G, H)$. In the above discussion we could have used, instead of $\mathbb{Z}$-coefficients, any commutative ring with a unit. However, the particular coefficients used are crucial in the results

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below, and integer coefficients are used in the proof of Theorem 2. If H is trivial, the rank of $H^0(G; E[G])$ is equal to $1 + \text{rank of } H^1(G; \mathbb{Z}G)$ and is called the number of ends of G. In this note, we show that a similar formula holds for certain types of pairs (G, H). The condition on (G, H) is similar to the planarity condition of [3], but different from it (see the example at the end). These lead to the calculation in Corollary 3 which was the original reason for looking at this situation. Kropholler and Roller obtained similar results by a different approach (see [5]). We prove in detail Proposition 1 which is typical of the results here.

Proposition 1. *Let G be a finitely generated group with one end and let H be an infinite cycle subgroup of G. Then the map*

$$i : H^1(G; \mathbb{Z}[G/H]) \to H^1(H; \mathbb{Z})$$

is trivial and

$$e(G, H) = 1 + \text{rank of } H^1(G; \mathbb{Z}[G/H]) .$$

Proof. Let $\tilde{K}$ be the Cayley complex of G with respect to a finite system of generators. Then $K = \tilde{K}/G$ is a finite wedge of circles. Let $\overline{K} = K/H$, $G' = \pi_1(K)$ and $\phi_1 : G' \to G$ the natural map. Let N be the kernel of ϕ_1, $H' = \phi_1^{-1}(H)$ and $\phi_2 = \phi_1|_{H'}$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^1(G; \mathbb{Z}[G/H]) & \xrightarrow{\;i\;} & H^1(H; \mathbb{Z}) \\ \downarrow{\scriptstyle \phi_1^*} & & \downarrow{\scriptstyle \phi_2^*} \\ H^1(G'; \mathbb{Z}[G'/H']) & \xrightarrow{\;i'\;} & H^1(H'; \mathbb{Z}) \end{array}$$

Let $\alpha : H \to \mathbb{Z}$ represent a generator of $H^1(H; \mathbb{Z})$. Since ϕ_2^* is injective, $\phi_2^*(\alpha) = \beta$ is non-trivial in $H^1(H'; \mathbb{Z})$ and is represented by $\alpha\phi_2 : \pi_1(\overline{K}) \to \mathbb{Z}$. To show that the map i is the zero map, it is enough to show that $m\beta$ is not in the image of i' for any non-zero integer m. Now, as in [2], we identify i' with the natural map $j : H^1_c(\overline{K}; \mathbb{Z}) \to H^1(\overline{K}; \mathbb{Z})$ where H^1_c denotes the cohomology with compact supports.

If $m\beta$ is the image of i', then there is a finite cocycle θ on $\overline{K}$ such that the induced map $\overline{\theta} : \pi_1(\overline{K}) \to Z$ obtained by evaluation of θ is equal to $m\beta$. Let L be a finite complex in $\overline{K}$ containing the support of θ and let X be a non-compact component of the closure $\overline{(\overline{K} - L)}$ of $(\overline{K} - L)$. Such a component exists since $\overline{K}$ is non-compact, which in turn follows from the assumption that the index of H in G is infinite. Let $Y = \overline{(\overline{K} - X)}$ and $B = X \cap Y = \text{frontier of } X = \text{frontier of } Y$. Since B is contained in L, we see that B is compact.

Let i_X denote the natural map $\pi_1(X) \to \pi_1(\overline{K})$ and let $q : \tilde{K} \to \overline{K}$ denote the covering projection. Since the support of θ is contained in Y, we have

$\overline{\theta}(i_X(\pi_1(X)))=0$, and therefore $m\alpha\phi_2 i_X(\pi_1(X))=0$. Since $m\neq 0$ it follows that $\phi_2 i_X(\pi_1(X))=0$. Hence $i_X(\pi_1(X))\subset N$ which is the kernel of ϕ_1 and of ϕ_2. This implies that $q^{-1}(X)$ contains a number of copies of X, say X_j indexed by $\mathbb{Z}$ such that $q|_{X_j}: X_j \to X$ is a homeomorphism for each j. Denoting by B_j the frontier of X_j, we see that for any j, $\tilde{K}-B_j$ has at least two infinite components which implies that the number of ends of $\tilde{K}$ is greater than one. This contradicts the assumption that G and hence $\tilde{K}$ has one end. Hence the map i is trivial and the proof of Proposition 1 is complete. □

The above argument goes through for any non-trivial map $\alpha: H\to\mathbb{Z}$, provided the cover corresponding to the kernel of α has one end. Thus we have the following theorem:

Theorem 2. *Let G be a finitely generated group and let H be a subgroup of infinite index in G such that for every non-trivial exact sequence*

$$\{e\}\to N\to H\to\mathbb{Z}\to\{0\}$$

the number of ends of the pair (G,N) is one. Then

$$e(G,H)=1+\text{rank of } H^1(G;\mathbb{Z}[G/H])\,. \qquad \square$$

Corollary 3. *Let $H\subset G$ be orientable Poincaré duality groups of dimensions $(n-1)$ and n respectively. Then $e(G,H)=2$.*

Proof. For any N as in Theorem 2, the cohomological dimension is $\leq n-2$ by [7]. Thus, $H^1(G;\mathbb{Z}[G/N])\cong H_{n-1}(G;\mathbb{Z}[G/N])\cong H_{n-1}(N;\mathbb{Z})$ (see [1]) is trivial. From the sequence (1) in the beginning, we have $e(G,H)=1+\text{rank of}$ $H^1(G;\mathbb{Z}[G/H])$. The latter group is isomorphic to $H_{n-1}(G;\mathbb{Z}[G/H])\cong H_{n-1}(H;\mathbb{Z})\cong\mathbb{Z}$. Hence $e(G,H)=2$. □

The following results were suggested by the referee:

Corollary 4. *Let $H\subset G$ be Poincaré duality groups of dimensions $(n-1)$ and n respectively. Then $e(G,H)=2$ if and only if the restriction to H of the dualising module for G is isomorphic to the dualising module for H.*

Proof. Let $D(G)$ and $D(H)$ denote the dualising modules for G and H; both are isomorphic to $\mathbb{Z}$ as additive groups. Then

$$\begin{aligned} H^1(G;\mathbb{Z}[G/H]) &\cong H_{n-1}(G;D(G)\otimes\mathbb{Z}[G/H])\cong H_{n-1}(H;D(G))\\ &\cong H^0(H,\operatorname{Hom}(D(H),D(G)))\,. \end{aligned}$$

The last group is isomorphic to $\mathrm{Hom}_H(D(H), D(G))$ which is non-zero if and only if $D(G)$ and $D(H)$ are isomorphic as H-modules. If G and H are orientable, the above reduces to Corollary 3. □

Example 5. The above results are not valid with $\mathbb{Z}_2$-coefficients (this was pointed out to us by C.H. Houghton and P.H. Kropholler) and thus assumptions above are not quite the planarity assumptions of [3]). The example is obtained by taking $G = \{a, b \mid bab^{-1} = a^{-1}\}$ and H to be the subgroup generated by b. The cover $\tilde{K}$ of K corresponding to H is homeomorphic to the interior of a Möbius band and therefore $e(G, H) = 1$. On the other hand, if our Proposition 1 were true with $\mathbb{Z}_2$-coefficients, by the analogue of Corollary 3, we would obtain $e(G, H) = 2$. However, we have the following:

Remark 6. With G and H as in Proposition 1, either $e(G, H) = 1 + \mathrm{rank}$ of $H^1(G; \mathbb{Z}_2[G/H])$ or there is a homomorphism $H \to \mathbb{Z}_2$ such that for the kernel N, $e(G, N) \geq 2$. A similar remark holds for $\mathbb{Z}_p$-coefficients.

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